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Distorted Heisenberg algebra and coherent states for isospectral oscillator Hamiltonians

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Abstract. The dynamical algebra associated with a family of isospectral oscillator Hamiltonians is studied through the analysis of its representation in the basis of energy eigenstates. It is shown that this representation becomes similar to that of the standard Heisenberg algebra, and it is dependent on a parameter $w \geqslant 0$. We call it the *distorted Heisenberg algebra*, where w is the distortion parameter. The corresponding coherent states for an arbitrary w are derived, and some particular examples are discussed in detail. A prescription to produce the squeezing, by adequately selecting the initial state of the system, is given.

1. Introduction

The well known coherent states of the harmonic oscillator turned out one of the most useful tools of quantum theory [1–3]. Introduced long ago by Schrödinger [4], they were later employed by Glauber and other authors in quantum optics [5–7]. Further developments of the subject made it possible to set up some specific definitions, applicable to various physical systems.

One possibility is to define the coherent states as eigenstates of an annihilation operator. Following this idea, the coherent states for a family of Hamiltonians isospectral to the harmonic oscillator were derived recently [8]. As there is a certain arbitrariness in the selection of the annihilation and creation operators for this system, the most obvious realization was chosen: the operators are adjoint to each other but their commutator is not the identity. In the same paper a different option of constructing the lowering and raising operators was also pursued: the creator was altered while the annihilator was not; they were no longer adjoint to each other, but their commutator was equal to the identity. This modified pair could, in principle, induce new coherent states, consistent with the application of a 'displacement operator' to the extremal state. However, the states thus derived turned out to be identical to the ones previously defined as eigenstates of the annihilator.

In the light of these results, it is interesting to pose the following questions: can both ideas be unified to yield lowering and raising operators which would be adjoint to each other and would commute to the identity, imitating the Heisenberg algebra? If so, what kind of coherent states would they generate?

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The goal of this paper is to find out the answers to these questions. In section 2 we will sketch the derivation of the family of Hamiltonians isospectral to the harmonic oscillator [8, 9]. Section 3 contains the construction of new pairs of annihilation and creation operators for these Hamiltonians; we will build these pairs from the generators of the standard Heisenberg algebra. Indeed, we will see that there is a family of such pairs depending on a parameter $w \geqslant 0$. In section 4 two sets of coherent states will be found for arbitrary values of w: the ones derivable as eigenstates of the annihilation operator and the ones resulting from the application of a 'displacement' operator on the extremal state. By fixing some specific values of w, we will attain three particularly interesting cases which will be discussed in section 5. We conclude with some general remarks in section 6.

2. The isospectral oscillator Hamiltonians H_λ

We are interested in a family of Hamiltonians H_λ which can be derived from the harmonic oscillator Hamiltonian H using a variant of the factorization method [9]. The standard factorization expresses H as two products

$$H = aa^\dagger - \tfrac{1}{2} \qquad H = a^\dagger a + \tfrac{1}{2} \tag{2.1}$$

where H and the annihilation a and creation $a^\dagger$ operators are given by

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{2} \qquad a = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right) \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x\right). \tag{2.2}$$

It can be proved that the first decomposition in (2.1) is not unique. Indeed, there exist more general operators b and $b^\dagger$ generating H:

$$H = bb^\dagger - \tfrac{1}{2} \qquad b = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right) \qquad b^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \beta(x)\right). \tag{2.3}$$

Hence, $\beta(x)$ obeys the Riccati equation $\beta' + \beta^2 = x^2 + 1$, whose general solution is

$$\beta(x) = x + \frac{\mathrm{e}^{-x^2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y} \qquad \lambda \in \mathbb{R}. \tag{2.4}$$

The inverted product $b^\dagger b$ is not related to H, but induces a different Hamiltonian

$$H_\lambda = b^\dagger b + \tfrac{1}{2} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\lambda(x) \tag{2.5}$$

$$V_\lambda(x) = \frac{x^2}{2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^{-x^2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}\right] \qquad |\lambda| > \frac{\sqrt{\pi}}{2}. \tag{2.6}$$

The Hamiltonians H and H_λ are connected by the following relation:

$$H_\lambda b^\dagger = b^\dagger(H + 1). \tag{2.7}$$

Therefore, if $|\psi_n\rangle$ are the standard eigenstates of H verifying $H|\psi_n\rangle = (n + \frac{1}{2})|\psi_n\rangle$, the states defined as

$$|\theta_n\rangle = \frac{b^\dagger|\psi_{n-1}\rangle}{\sqrt{n}} \qquad n = 1, 2, 3, \ldots \tag{2.8}$$

are normalized orthogonal eigenstates of H_λ with eigenvalues $E_n = n + \frac{1}{2}$, respectively. The ground state of H_λ is disconnected from the other eigenstates, it has eigenvalue $E_0 = \frac{1}{2}$ and satisfies $b|\theta_0\rangle = 0$. In the coordinate representation it is given by

$$\theta_0(x) \propto \frac{\mathrm{e}^{-x^2/2}}{\lambda + \int_0^x \mathrm{e}^{-y^2}\,\mathrm{d}y}. \tag{2.9}$$

Summarizing this section, $\{H_\lambda,\ |\lambda| > \sqrt{\pi}/2\}$ represents a family of Hamiltonians with the same spectrum as the harmonic oscillator. The relations necessary to set up the creation and annihilation operators of H_λ are

$$\begin{aligned} b|\theta_n\rangle &= \sqrt{n}\,|\psi_{n-1}\rangle \qquad & b^\dagger|\psi_n\rangle &= \sqrt{n+1}\,|\theta_{n+1}\rangle \\ a|\psi_n\rangle &= \sqrt{n}\,|\psi_{n-1}\rangle \qquad & a^\dagger|\psi_n\rangle &= \sqrt{n+1}\,|\psi_{n+1}\rangle\,. \end{aligned} \tag{2.10}$$

3. Distorted Heisenberg algebra of H_λ

It is important to identify now a suitable pair of annihilation and creation operators for H_λ. The obvious choice follows immediately from (2.10) [9, 8]:

$$A = b^\dagger a b \qquad A^\dagger = b^\dagger a^\dagger b\,. \tag{3.1}$$

The effective action of, let us say, the annihilation operator A comes after three intermediate transformations: we take an eigenstate $|\theta_n\rangle$ of H_λ and transform it, by the action of b, in $|\psi_{n-1}\rangle$, an eigenstate of H; then, a transforms $|\psi_{n-1}\rangle$ in $|\psi_{n-2}\rangle$; finally, $|\theta_{n-1}\rangle$ is obtained through the action of $b^\dagger$ on $|\psi_{n-2}\rangle$. A similar procedure works for $A^\dagger$.

As it is easily seen, the operators defined in (3.1) are reciprocally adjoint, but they do not commute to the identity. Different annihilation and creation operators arise if A is left unchanged but we define a new creator $B^\dagger$, with the requirement $[A, B^\dagger] = 1$. The operator $B^\dagger$ turns out to be [8]

$$B^\dagger = b^\dagger\, a^\dagger\, \frac{1}{(N+1)(N+2)}\, b \tag{3.2}$$

where $N = a^\dagger a$ is the standard number operator. Obviously, $B^\dagger$ is not the adjoint of A. A third realization, and this is one of the results of this paper, arises when both A and $A^\dagger$ are substituted by new annihilation and creation operators C and $C^\dagger$ chosen to be reciprocally adjoint, and such that their commutator is the identity on a subspace $\mathcal{H}_s$ of the state space $\mathcal{H}$:

$$[C, C^\dagger] = 1 \qquad \text{on} \quad \mathcal{H}_s \subset \mathcal{H}\,. \tag{3.3}$$

In the spirit of [8], we propose

$$C = b^\dagger f(N) a b \qquad C^\dagger = b^\dagger a^\dagger f(N) b \tag{3.4}$$

where $f(x)$ is a real function to be determined. Taking into account (2.10), $[C, C^\dagger]$ acts on $|\theta_n\rangle$ as follows:

$$[C, C^\dagger]|\theta_n\rangle = \left\{n^2(n+1)[f(n-1)]^2 - (n-1)^2 n[f(n-2)]^2\right\}|\theta_n\rangle\,. \tag{3.5}$$

For $n = 0$ and $n = 1$ we get

$$[C, C^\dagger]|\theta_0\rangle = 0 \qquad [C, C^\dagger]|\theta_1\rangle = 2[f(0)]^2|\theta_1\rangle\,. \tag{3.6}$$

We now impose (3.3), with $\mathcal{H}_s$ the subspace generated by $\{|\theta_n\rangle; n \geqslant 2\}$. Defining a new function, $g(x) = (x+1)^2(x+2)[f(x)]^2,\ x \in \mathbb{N}$, it has to verify the difference equation

$$g(x+1) - g(x) = 1 \tag{3.7}$$

whose general solution is $g(x) = x + w(x)$, being $w(x)$ an arbitrary periodic function, with period equal to one. Therefore, $f(x)$ takes the form

$$f(x) = \frac{1}{x+1}\sqrt{\frac{x+w(x)}{x+2}}\,. \tag{3.8}$$

The function $f(x)$ must be real, hence, $x + w(x) \geqslant 0$, $\forall\ x \in \mathbb{N}$. This fact, and the periodicity of $w(x)$ imply that the relevant value of $w(x)$ is $w(0)$ with

$$w \equiv w(0) \geqslant 0 . \tag{3.9}$$

The form of the operators C and $C^\dagger$ satisfying (3.3) is then

$$C_w = b^\dagger \frac{1}{N+1}\sqrt{\frac{N+w(N)}{N+2}}ab \qquad C_w^\dagger = b^\dagger a^\dagger \frac{1}{N+1}\sqrt{\frac{N+w(N)}{N+2}}b \tag{3.10}$$

where the subindex labels the dependence of C and $C^\dagger$ on the parameter w. The action of C_w, $C_w^\dagger$ and $[C_w, C_w^\dagger]$ on $\{|\theta_n\rangle,\ n \in \mathbb{N}\}$ is

$$C_w|\theta_n\rangle = (1-\delta_{n,0}-\delta_{n,1})\ \sqrt{n-2+w}\ |\theta_{n-1}\rangle \tag{3.11}$$

$$C_w^\dagger|\theta_n\rangle = (1-\delta_{n,0})\ \sqrt{n-1+w}\ |\theta_{n+1}\rangle \tag{3.12}$$

$$[C_w, C_w^\dagger]|\theta_n\rangle = [1-\delta_{n,0}+\delta_{n,1}(w-1)]|\theta_n\rangle = \begin{cases} 0 & n=0 \\ w|\theta_1\rangle & n=1 \\ |\theta_n\rangle & n \geqslant 2 . \end{cases} \tag{3.13}$$

As this action resembles that of the generators of the Heisenberg algebra a, $a^\dagger$ and $[a, a^\dagger]$ on the harmonic oscillator basis $\{|\psi_n\rangle\}$, we are led to define the two operators

$$X_w = \frac{C_w^\dagger + C_w}{\sqrt{2}} \qquad P_w = \mathrm{i}\ \frac{C_w^\dagger - C_w}{\sqrt{2}} . \tag{3.14}$$

They are to H_λ as the usual coordinate X and momentum P operators are to the harmonic oscillator Hamiltonian. The commutator of X_w and P_w in the basis $\{|\theta_n\rangle,\ n \in \mathbb{N}\}$ is

$$[X_w, P_w]|\theta_n\rangle = \mathrm{i}[1-\delta_{n,0}+\delta_{n,1}(w-1)]|\theta_n\rangle = \begin{cases} 0 & n=0 \\ \mathrm{i}w|\theta_1\rangle & n=1 \\ \mathrm{i}|\theta_n\rangle & n \geqslant 2 . \end{cases} \tag{3.15}$$

From equations (3.11)–(3.15), it can be seen that the representation of C_w, $C_w^\dagger$, X_w and P_w on the basis $\{|\theta_n\rangle,\ n \in \mathbb{N}\}$ is reducible because there are two invariant subspaces, one of them generated by $|\theta_0\rangle$ and the other one by $\{|\theta_n\rangle, n \geqslant 1\}$. We denote them as $\mathcal{H}_0$ and $\mathcal{H}_r$, respectively. In $\mathcal{H}_0$ all the operators C_w, $C_w^\dagger$, X_w and P_w are trivially represented by the 1×1 null matrix. The relevant representation for those operators arises when we consider their action on vectors $|\psi\rangle \in \mathcal{H}_r$. This representation is similar to the one of the standard Heisenberg algebra, however, it depends on the parameter w. This makes the difference between the two representations compared here. Thus, we call the 'distorted Heisenberg algebra' the algebra generated by C_w and $C_w^\dagger$ (or by X_w and P_w). One reason to choose this name is because the representation of C_w and $C_w^\dagger$ on $\mathcal{H}_r$ can be thought of as coming from that of a and $a^\dagger$ on $\mathcal{H}$ after two steps of distortion: first, we remove the ground state of the oscillator Hamiltonian, second, we deform the representation induced by the remaining basis vectors through the introduction of a distortion parameter w. However, it is important to recall that C_w, $C_w^\dagger$, X_w and P_w are not simple generalizations of a, $a^\dagger$, X and P, in the sense that it is impossible to get the action of the last ones on $\mathcal{H}$ as a limit procedure, for w tending to a specific value, of the action of the first ones on $\mathcal{H}_r$. We postpone to section 5 the discussion of cases, for particular values of w, for which the distorted Heisenberg algebra is the closest to the Heisenberg algebra on $\mathcal{H}$. This will give a better support to our terminology. Meanwhile, here and in the next section we derive the general results, valid for the full range $w \geqslant 0$.

4. New coherent states of H_λ

It is well known that, for a general system, there are three non-equivalent definitions for the coherent states [1–3]. One consists of defining them as eigenstates of the annihilation operator of the system, denoted here by J. Another possibility is to define them as the vectors resulting from the application of the unitary displacement operator $\exp(zJ^\dagger - \bar{z}J)$ on an extremal state $|\varphi_0\rangle$, which usually is an eigenstate of J with zero eigenvalue, i.e. $J|\varphi_0\rangle = 0$ (here $z \in \mathbb{C}$, the bar over z means complex conjugation, and $J^\dagger$ denotes the adjoint of J). A third definition characterizes the coherent states as minimum-uncertainty states (see also [10]). In [8] a set of coherent states for H_λ was derived using the first definition, with the annihilation operator A given by (3.1). An additional set of coherent states was found through the second definition, but employing the non-unitary displacement operator $\exp(zB^\dagger - \bar{z}A)$, the extremal state $|\theta_1\rangle$, and the operator $B^\dagger$ given in (3.2). The two sets turned out to be equal [8].

In this section, new coherent states of H_λ will be constructed using the first definition and a modified version of the second one departing from the annihilation and creation operators given in (3.10). In both cases, the uncertainty product of the distorted position and momentum operators of (3.14) on the resultant states will be found in order to compare our new coherent states and the standard ones, which minimize the uncertainty product $(\Delta X)(\Delta P)$.

4.1. Coherent states as eigenstates of C_w

Let us denote the coherent states $|z, w\rangle$, to show explicitly their dependence on the parameter w. They are eigenstates of C_w:

$$C_w|z, w\rangle = z|z, w\rangle \qquad z \in \mathbb{C}. \tag{4.1}$$

To have their explicit form, we decompose $|z, w\rangle$ in terms of the basis $|\theta_n\rangle$:

$$|z, w\rangle = \sum_{n=0}^{\infty} a_n \, |\theta_n\rangle. \tag{4.2}$$

Substituting this expression in (4.1) and using (3.11) we get the coefficients a_n. If we suppose that $w \neq 0$, we get

$$a_0 = 0 \qquad a_{n+1} = \frac{\sqrt{\Gamma(w)}\, z^n}{\sqrt{\Gamma(w+n)}} a_1 \qquad n \in \mathbb{N}. \tag{4.3}$$

If we chose $a_1 \geqslant 0$, the normalization condition leads to

$$|z, w\rangle = \sqrt{\frac{\Gamma(w)}{{}_1F_1(1, w; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(w+n)}} \, |\theta_{n+1}\rangle \qquad z = re^{i\varphi} \tag{4.4}$$

where ${}_1F_1(a, b; x)$ is the hypergeometric function

$${}_1F_1(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{x^k}{k!} \tag{4.5}$$

$r = |z| \in \mathbb{R}$, and $\varphi \in \mathbb{R}$. We again realize, as in [8], that $z = 0$ is doubly degenerated with eigenkets $|\theta_0\rangle$ and $|\theta_1\rangle$.

Observe that, although the case $w = 0$ is excluded of (4.3), the states $|z, w\rangle$ of (4.4) tend to a well defined limit when $w \to 0$

$$|z, 0\rangle = e^{i\varphi} e^{-r^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, |\theta_{n+2}\rangle. \tag{4.6}$$

We have checked this result by performing a direct calculation, similar as the previous one, but taking $w = 0$ from the very beginning, which led us to the same states (4.6) (modulo a phase). These will be considered again in section 5.

Let us analyse the completeness of the set $\{|\theta_0\rangle, |z, w\rangle; z \in \mathbb{C}\}$. We impose

$$I = |\theta_0\rangle\langle\theta_0| + \int |z, w\rangle\langle z, w| \, \mathrm{d}\mu(z, w)$$
$$= |\theta_0\rangle\langle\theta_0| + \int |z, w\rangle\langle z, w| \, \sigma(r, w) \, r \, \mathrm{d}r \, \mathrm{d}\varphi \tag{4.7}$$

where $\mathrm{d}\mu(z, w)$ is the unknown measure. Following a standard procedure [11, 12] one finds

$$\sigma(r, w) = \frac{{}_1F_1(1, w; r^2)}{\pi\Gamma(w)} \, \mathrm{e}^{-r^2} r^{2(w-1)} . \tag{4.8}$$

Here, $\sigma(r, w)$ is simpler than the corresponding function obtained in [8]. It is possible to express any coherent states $|z', w\rangle$ in terms of the others:

$$|z', w\rangle = \int |z, w\rangle\langle z, w|z', w\rangle \, \mathrm{d}\mu(z, w) \qquad |z', w\rangle \neq |\theta_0\rangle \tag{4.9}$$

with a kernel given by

$$\langle z, w|z', w\rangle = \frac{{}_1F_1(1, w; \bar{z}z')}{\sqrt{{}_1F_1(1, w; r^2) \; {}_1F_1(1, w; r'^2)}} . \tag{4.10}$$

Using equation (4.7), any element $|h\rangle \in \mathcal{H}$ can be expanded in terms of the coherent states as

$$|h\rangle = h_0|\theta_0\rangle + \int \tilde{h}(z, \bar{z}, w)|z, w\rangle \, \mathrm{d}\mu(z, w) \tag{4.11}$$

where $h_0 \equiv \langle\theta_0|h\rangle$, and

$$\tilde{h}(z, \bar{z}, w) \equiv \langle z, w|h\rangle = \sqrt{\frac{\Gamma(w)}{{}_1F_1(1, w; r^2)}} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{\Gamma(w+n)}} \langle\theta_{n+1}|h\rangle . \tag{4.12}$$

The time evolution of $|z, w\rangle$ is quite simple:

$$U(t)|z, w\rangle = \sqrt{\frac{\Gamma(w)}{{}_1F_1(1, w; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(w+n)}} \mathrm{e}^{-\mathrm{i}tH_\lambda} |\theta_{n+1}\rangle = \mathrm{e}^{-\mathrm{i}3t/2} |z(t), w\rangle \tag{4.13}$$

where $z(t) \equiv z\,\mathrm{e}^{-\mathrm{i}t}$, and $U(t)$ is the evolution operator of the system from 0 to t.

The mean value and the uncertainty of an operator K in the coherent state $|z, w\rangle$ are denoted by

$$\langle K\rangle \equiv \langle z, w|K|z, w\rangle \qquad \Delta K \equiv \sqrt{\langle K^2\rangle - \langle K\rangle^2} . \tag{4.14}$$

For the Hamiltonian H_λ we get

$$\langle H_\lambda\rangle = \tfrac{1}{2} + \frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} . \tag{4.15}$$

By the previous construction, $\langle C_w\rangle = z$ and $\langle C_w^\dagger\rangle = \bar{z}$. Therefore, $\langle X_w\rangle$ and $\langle P_w\rangle$ become

$$\langle X_w\rangle = \frac{\langle C_w^\dagger\rangle + \langle C_w\rangle}{\sqrt{2}} = \frac{\bar{z} + z}{\sqrt{2}} \tag{4.16}$$

$$\langle P_w\rangle = \mathrm{i}\frac{\langle C_w^\dagger\rangle - \langle C_w\rangle}{\sqrt{2}} = \mathrm{i}\frac{\bar{z} - z}{\sqrt{2}} . \tag{4.17}$$

They are equal to the corresponding harmonic oscillator results. Let us now calculate

$$\langle C_w C_w^\dagger + C_w^\dagger C_w \rangle = w - 1 + r^2 + \frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} \qquad \langle C_w^2 \rangle = \overline{\langle C_w^{\dagger 2} \rangle} = z^2 . \tag{4.18}$$

Hence,

$$\langle X_w^2 \rangle = \tfrac{1}{2}(z^2 + \bar{z}^2) + \frac{w-1}{2} + \frac{1}{2}\left(r^2 + \frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} \right) \tag{4.19}$$

$$\langle P_w^2 \rangle = -\tfrac{1}{2}(z^2 + \bar{z}^2) + \frac{w-1}{2} + \frac{1}{2}\left(r^2 + \frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} \right) . \tag{4.20}$$

Thus, the uncertainties of X_w and P_w, and their product, are

$$(\Delta X_w)^2 = (\Delta P_w)^2 = (\Delta X_w)(\Delta P_w) = \frac{1}{2}\left(w - 1 - r^2 + \frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} \right) . \tag{4.21}$$

Notice that the uncertainty relation (4.21) has radial symmetry on the complex plane of z. A plot of it as a function of $r = |z|$ for different values of w is shown in figure 1. As we can see, $w/2 \leqslant (\Delta X_w)(\Delta P_w) \leqslant \frac{1}{2}$ if $0 \leqslant w \leqslant 1$ and $\frac{1}{2} \leqslant (\Delta X_w)(\Delta P_w) \leqslant w/2$ if $w \geqslant 1$. Thus, the coherent states just derived are close to the minimum uncertainty ones for large values of r, and also for small values of w. In section 5 we will find explicit values of w for which our coherent states become minimum uncertainty states satisfying $(\Delta X_w)(\Delta P_w) = \frac{1}{2}$ for any z.

One question arises naturally: what happens in the harmonic oscillator limit? This can be answered if we realize that, in the limit $|\lambda| \to \infty$, $H_\lambda \to H$. Moreover, in this limit

$$b \to a \qquad b^\dagger \to a^\dagger \qquad |\theta_n\rangle \to |\psi_n\rangle . \tag{4.22}$$

Therefore, the corresponding limits for C_w and $C_w^\dagger$ are

$$C_{w,0} \equiv \lim_{|\lambda| \to \infty} C_w = a^\dagger \frac{1}{N+1} \sqrt{\frac{N + w(N)}{N+2}}\, a^2 \tag{4.23}$$

$$C_{w,0}^\dagger \equiv \lim_{|\lambda| \to \infty} C_w^\dagger = a^{\dagger 2} \frac{1}{N+1} \sqrt{\frac{N + w(N)}{N+2}}\, a . \tag{4.24}$$

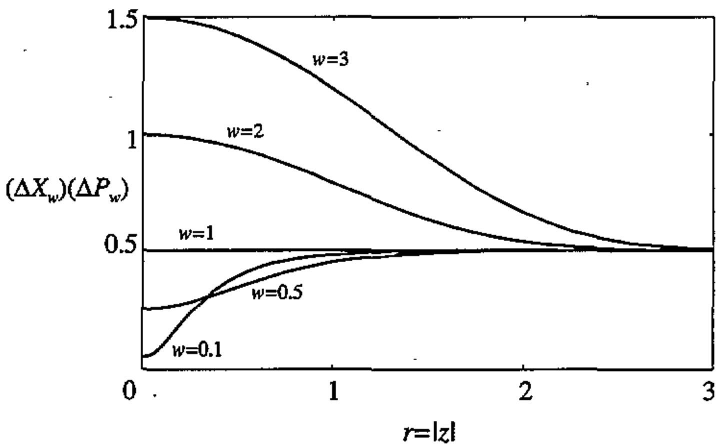

Figure 1. Plot of $(\Delta X_w)(\Delta P_w)$ as a function of $r = |z|$ for different values of w.

For the coherent states we have

$$|z, w\rangle_0 \equiv \lim_{|\lambda|\to\infty} |z, w\rangle = \sqrt{\frac{\Gamma(w)}{{}_1F_1(1, w; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(w+n)}} |\psi_{n+1}\rangle . \tag{4.25}$$

We see that, in general, the coherent states derived here are different from the standard ones of H, even though $H_\lambda \to H$ when $|\lambda| \to \infty$. In section 5, we will analyse other limit cases, by approaching specific values of w, which will provide us with more insight about the differences and similarities of our coherent states and the standard ones.

4.2. Displacement operator technique and coherent states

According to the second definition, the coherent states we should now find would result from the application of the displacement operator $D(z) = \exp(zC_w^\dagger - \bar{z}C_w)$ on an extremal state $|\varphi_0\rangle$ which obeys $C_w|\varphi_0\rangle = 0$. For H_λ and C_w given by (2.5), (2.6) and (3.10) there are two extremal states $|\theta_0\rangle$ and $|\theta_1\rangle$:

$$C_w|\theta_1\rangle = C_w|\theta_0\rangle = 0. \tag{4.26}$$

If $|\theta_0\rangle$ is taken, we will not obtain any additional coherent states because $C_w^\dagger|\theta_0\rangle = 0$, which implies that $|\theta_0\rangle$ is invariant under the application of $D(z)$. The only non-trivial possibility is to take $|\varphi_0\rangle = |\theta_1\rangle$. However, the way in which $[C_w, C_w^\dagger]$ acts on the basis vectors $|\theta_n\rangle$ (see equation (3.13)) disables the factorization of $D(z)$ to simplify the calculation of $D(z)|\theta_1\rangle$. Therefore, we decided to consider the non-unitary operator

$$D_w(z) \equiv \mathrm{e}^{zC_w^\dagger} \tag{4.27}$$

and look for the states of the form

$$|z, w\rangle_d \propto D_w(z)|\theta_1\rangle . \tag{4.28}$$

Using equation (3.12), we obtain

$$|z, w\rangle_d = \frac{1}{\sqrt{\Gamma(w)\ {}_1F_1(w, 1; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sqrt{\Gamma(w+n)}\, |\theta_{n+1}\rangle . \tag{4.29}$$

Notice that, for $w = 0$, we have $|z, 0\rangle_d = |\theta_1\rangle$, and there is no family of coherent states.

The completeness of this new set, $\{|\theta_0\rangle, |z, w\rangle_d; z \in \mathbb{C}\}$, now reads

$$\begin{aligned} I &= |\theta_0\rangle\langle\theta_0| + \int |z, w\rangle_{d\ d}\langle z, w|\, \mathrm{d}\mu_d(z, w) \\ &= |\theta_0\rangle\langle\theta_0| + \int |z, w\rangle_{d\ d}\langle z, w|\, \sigma_d(r, w)\, r\, \mathrm{d}r\, \mathrm{d}\varphi . \end{aligned} \tag{4.30}$$

As the relevant values of w are $w \geqslant 0$, we can define $\sigma_d(r, w) = \Gamma(w)\, {}_1F_1(w, 1; r^2) \times\eta(r^2, w)$, and following a procedure similar to that of [11, 12], it turns out that the function $\eta(x, w)$ must satisfy

$$\int_0^\infty \eta(x, w)\, x^{m-1}\, \mathrm{d}x = \frac{(\Gamma(m))^2}{\pi\, \Gamma(m+w-1)} \qquad m = 1, 2, \ldots . \tag{4.31}$$

We have to solve a typical 'momentum problem' (see [13] and references quoted therein). To do it, we can use the Mellin transform technique, as we did to find $\sigma(r, w)$ in (4.8), and we get the following result:

$$\eta(x,w)=\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{x^l}{(l!)^2\,\Gamma(w-l-1)}[-\ln x+2\psi(l+1)-\psi(w-l-1)] \qquad w\notin\mathbb{N}$$

$$\eta(x,n)=\frac{1}{\pi}\sum_{l=0}^{n-2}\frac{x^l}{(l!)^2\,(n-l-2)!}[-\ln x+2\psi(l+1)-\psi(n-l-1)] \tag{4.32}$$

$$+\frac{x^{n-1}}{\pi[(n-1)!]^2}\,{}_2F_2(1,1;n,n;-x) \qquad n=1,2,3\ldots$$

where $\psi(y)=[\Gamma(y)]^{-1}\,\mathrm{d}\Gamma(y)/\mathrm{d}y$, and ${}_2F_2(1,1;n,n;x)$ is a generalized hypergeometric function [14]. In the last equation, if $n=1$ the sum does not appear, and the generalized hypergeometric function is very simple; the result is

$$\eta(x,1)=\mathrm{e}^{-x}/\pi\,.$$

In the case we are considering, the reproducing kernel is

$${}_d\langle z,w|z',w\rangle_d=\frac{{}_1F_1(w,1;\bar{z}z')}{\sqrt{{}_1F_1(w,1;r^2)\;{}_1F_1(w,1;r'^2)}}\,. \tag{4.33}$$

The time evolution of these states is equal to that of (4.13), $U(t)|z,w\rangle_d=\mathrm{e}^{-\mathrm{i}3t/2}|z(t),w\rangle_d$. The mean value is defined as usual. Hence

$$\langle H_\lambda\rangle_d=\tfrac{3}{2}+r^2\,S(w,r^2) \qquad \langle C_w\rangle_d=\overline{\langle C_w^\dagger\rangle_d}=z\,S(w,r^2) \tag{4.34}$$

where

$$S(w,r^2)=w\,\frac{{}_1F_1(w+1,2;r^2)}{{}_1F_1(w,1;r^2)}\,. \tag{4.35}$$

From equation (3.14) one gets

$$\langle X_w\rangle_d=\frac{(\bar{z}+z)}{\sqrt{2}}\,S(w,r^2) \qquad \langle P_w\rangle_d=\frac{\mathrm{i}(\bar{z}-z)}{\sqrt{2}}\,S(w,r^2)\,. \tag{4.36}$$

In order to obtain $\langle X_w^2\rangle_d$ and $\langle P_w^2\rangle_d$, we find first

$$\langle C_w^2\rangle_d=\overline{\langle C_w^{\dagger 2}\rangle_d}=z^2\,T(w,r^2) \tag{4.37}$$

$$\langle C_wC_w^\dagger+C_w^\dagger C_w\rangle_d=-1+\frac{1-w+2w\,{}_1F_1(w+1,1;r^2)}{{}_1F_1(w,1;r^2)} \tag{4.38}$$

where we have introduced the function $T(w,r^2)$, defined as

$$T(w,r^2)=\frac{w(w+1)}{2}\,\frac{{}_1F_1(w+2,3;r^2)}{{}_1F_1(w,1;r^2)}\,. \tag{4.39}$$

Therefore,

$$\langle X_w^2\rangle_d=-\tfrac{1}{2}+\tfrac{1}{2}(\bar{z}^2+z^2)\,T(w,r^2)+\frac{1-w+2w\,{}_1F_1(w+1,1;r^2)}{2\,{}_1F_1(w,1;r^2)} \tag{4.40}$$

$$\langle P_w^2\rangle_d=-\tfrac{1}{2}-\tfrac{1}{2}(\bar{z}^2+z^2)\,T(w,r^2)+\frac{1-w+2w\,{}_1F_1(w+1,1;r^2)}{2\,{}_1F_1(w,1;r^2)}\,. \tag{4.41}$$

Now, it is very easy to get the uncertainties $(\Delta X_w)_d$, $(\Delta P_w)_d$:

$$(\Delta X_w)_d^2 = \frac{(\bar{z}+z)^2}{2}\left[T(w,r^2) - S^2(w,r^2)\right] - r^2\,T(w,r^2) \\ -\tfrac{1}{2} + \frac{1-w+2w\,{}_1F_1(w+1,1;r^2)}{2\,{}_1F_1(w,1;r^2)} \tag{4.42}$$

$$(\Delta P_w)_d^2 = -\frac{(\bar{z}-z)^2}{2}\left[T(w,r^2) - S^2(w,r^2)\right] - r^2\,T(w,r^2) \\ -\tfrac{1}{2} + \frac{1-w+2w\,{}_1F_1(w+1,1;r^2)}{2\,{}_1F_1(w,1;r^2)}. \tag{4.43}$$

From (4.42) and (4.43) it is clear that, in contrast to the previous case, $(\Delta X_w)_d$, $(\Delta P_w)_d$ and their product do not have radial symmetry on the complex plane. Their dependence on $\varphi = \arg(z)$ means that, even though the evolution of a coherent state of kind $|z, w\rangle_d$ is equal to the one of a coherent state of kind $|z, w\rangle$, the uncertainties of X_w and P_w change in time for the states $|z, w\rangle_d$ but remain static for $|z, w\rangle$ (see equation (4.21)). This immediately suggests a very interesting use of the coherent states derived in this section: let us fix the values of w and $r = |z|$ and let us take as initial condition one of the states $|z, w\rangle_d$ having a maximum value of $(\Delta X_w)_d$. From figure 2 is is clear that this occurs for $\varphi = 0$ or $\varphi = \pi$ if $0 < w < 1$ and for $\varphi = \pi/2$ or $\varphi = 3\pi/2$ if $w > 1$; this can also be proved analytically. Now, let us evolve this initial state by a time $t = T/4$, where T is the period of the potentials (2.6), which in the units we are using becomes $T = 2\pi$. At the end of this interval the initial state has evolved into a different coherent state, $|z\,e^{-i\pi/2}, w\rangle_d$, and the uncertainty $(\Delta X_w)_d$ will be minimum. This is nothing but a maximum efficiency squeezing operation on the initial coherent state. The point is that we did not have to apply any sophisticated sequence of external potentials on our system to induce the squeezing operation. If we just select an adequate coherent state $|z, w\rangle_d$ as the initial condition, the

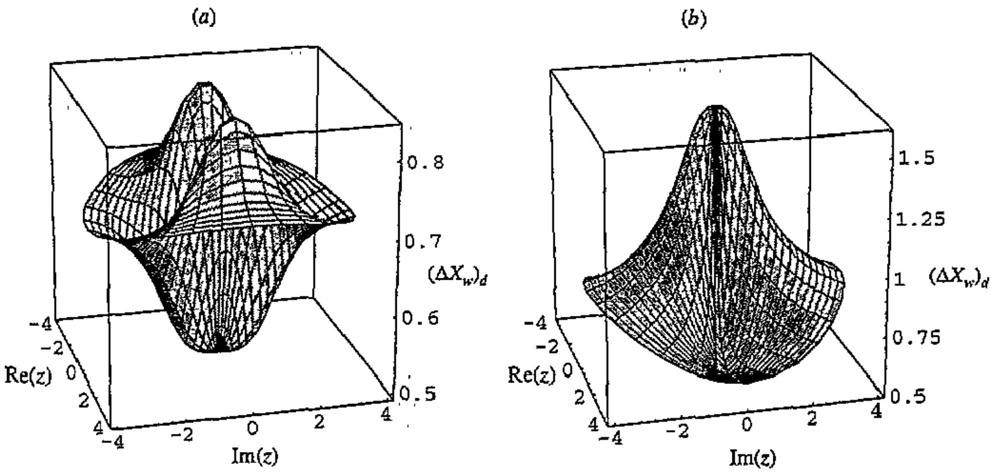

Figure 2. Plot of $(\Delta X_w)_d$ as a function of z for two w-values: (*a*) $w = 0.5$ and (*b*) $w = 5$. Notice that, for fixed values of $r = |z|$ and w, the maximum of $(\Delta X_w)_d$ occurs for $\varphi = \arg(z) = 0$ or π if $0 < w < 1$, while it occurs for $\varphi = \pi/2$ or $3\pi/2$ if $w > 1$. The corresponding graphs for $(\Delta P_w)_d$ can be obtained from the previous ones by a rotation of $\pi/2$ around the vertical axis.

evolution does the work. Of course, it is possible to design a scheme aimed to produce the inverse process, i.e. the maximum efficiency expansion operation. It is up to the designer to select which of these processes he is interested in.

Finally, the harmonic oscillator limit of these states is

$$|z, w\rangle_{d0} \equiv \lim_{|\lambda|\to\infty} |z, w\rangle_d = \frac{1}{\sqrt{\Gamma(w)\ {}_1F_1(w, 1; r^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sqrt{\Gamma(w+n)}\ |\psi_{n+1}\rangle\,. \tag{4.44}$$

Once again, we notice that $|z, w\rangle_{d0}$ does not coincide, in general, with a standard coherent state.

5. Particular cases

By taking three specific values of w, we now analyse particular situations for which our previous formulae take a simpler form. We will study the cases with $w = 0$, $w = 1$ and $w = 2$.

5.1. The case $w = 1$

Here, the subspace $\mathcal{H}_r$, which is invariant under C_1 and $C_1^\dagger$, also acquires the property (3.3) of $\mathcal{H}_s$. If we restrict the action of C_1 and $C_1^\dagger$ to $\mathcal{H}_r$, we then get a slight modification to the standard representation of the Heisenberg algebra

$$C_1|\theta_n\rangle = \sqrt{n-1}\,|\theta_{n-1}\rangle \qquad C_1^\dagger|\theta_n\rangle = \sqrt{n}\ |\theta_{n+1}\rangle \qquad [C_1, C_1^\dagger]|\theta_n\rangle = |\theta_n\rangle \qquad n \geqslant 1\,. \tag{5.1}$$

The two sets of coherent states derived in section 4 become equal:

$$|z, 1\rangle = |z, 1\rangle_d = \mathrm{e}^{-r^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |\theta_{n+1}\rangle\,. \tag{5.2}$$

They are the standard coherent states if we relabel the eigenstates of H_λ as $|\varphi_n\rangle \equiv |\theta_{n+1}\rangle$. The measure functions $\sigma(r, 1)$, $\sigma_d(r, 1)$, and the kernels (4.10) and (4.33) are transformed into the standard ones:

$$\sigma(r, 1) = \sigma_d(r, 1) = \frac{1}{\pi} \qquad \langle z, 1|z', 1\rangle = {}_d\langle z, 1|z', 1\rangle_d = \exp\left(-\frac{r^2}{2} - \frac{r'^2}{2} + \bar{z}z'\right)\,. \tag{5.3}$$

Due to the fact that the state on $\mathcal{H}_r$ with the minimum value of the energy is $|\theta_1\rangle$, $\langle H_\lambda\rangle$ becomes slightly different to the standard result:

$$\langle H_\lambda\rangle = \langle H_\lambda\rangle_d = \tfrac{3}{2} + r^2\,. \tag{5.4}$$

However, those coherent states are minimum uncertainty states, as they verify

$$(\Delta X_1)(\Delta P_1) = (\Delta X_1)_d(\Delta P_1)_d = \tfrac{1}{2}\,. \tag{5.5}$$

This result justifies, once again, the name selected for the algebra generated by C_w and $C_w^\dagger$, because we have found one w-value for which it reduces to the standard Heisenberg algebra on $\mathcal{H}_r \subset \mathcal{H}$. We will see next that $w = 1$ is not the only value inducing this behaviour.

5.2. The case $w = 0$

Let us take the limit $w \to 0$ in all formulae of sections 3 and 4. The subspace $\mathcal{H}_r$ decomposes now into two invariant subspaces: one of them is generated by $|\theta_1\rangle$ while the other one is $\mathcal{H}_s$, generated by $\{|\theta_n\rangle,\ n \geqslant 2\}$. The relevant representation of C_0 and $C_0^\dagger$ arises from the restriction of these operators to $\mathcal{H}_s$. We get again a slight modification of the standard Heisenberg algebra representation:

$$\begin{aligned} &C_0|\theta_n\rangle = \sqrt{n-2}\ |\theta_{n-1}\rangle \\ &C_0^\dagger|\theta_n\rangle = \sqrt{n-1}\ |\theta_{n+1}\rangle \quad [C_0, C_0^\dagger]|\theta_n\rangle = |\theta_n\rangle \qquad n \geqslant 2\,. \end{aligned} \tag{5.6}$$

The coherent states which are eigenstates of C_0, denoted as $|z,0\rangle$, are given in (4.6). However, those arising from the action of $D_0(z) = \exp(zC_0^\dagger)$, denoted as $|z,0\rangle_d$, cannot be found from (4.29) because those were derived taking $|\theta_1\rangle$ as extremal state, but now it does not belong to $\mathcal{H}_s$. Here, the extremal state inducing non-trivial coherent states is $|\theta_2\rangle$. A similar calculation as that of subsection 4.2 leads immediately to $|z,0\rangle_d$. These states, modulo a phase, are equal to those obtained in (4.6):

$$|z,0\rangle_d = \mathrm{e}^{-\mathrm{i}\varphi}|z,0\rangle = \mathrm{e}^{-r^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|\theta_{n+2}\rangle\,. \tag{5.7}$$

They are again as the standard coherent states. The measure function (4.8) and the kernel (4.10) are equal (modulo a phase) to the standard ones, and to those of the previous section (see equation (5.3)). The mean value of H_λ is different because we depart from a different extremal state:

$$\langle H_\lambda\rangle = \langle H_\lambda\rangle_d = \tfrac{5}{2} + r^2\,. \tag{5.8}$$

However, once again we find that $|z,0\rangle$ are minimum uncertainty states:

$$(\Delta X_0)(\Delta P_0) = (\Delta X_0)_d(\Delta P_0)_d = \tfrac{1}{2}\,. \tag{5.9}$$

Thus, we have found some additional information which we did not foresee before: through the analysis of the coherent states resulting from the two definitions considered in section 4, we have been able to construct the coherent states characteristic of the third definition. We will next analyse the simplest particular case involving a representation qualitatively different from the standard Heisenberg algebra representation.

5.3. The case $w = 2$

Let us put $w = 2$ in all the relationships of sections 3 and 4. As in the general case of Section 3, the relevant subspace is $\mathcal{H}_r$, generated by the basis $\{|\theta_n\rangle, n \geqslant 1\}$. However, unlike the two previous particular cases, we do not now obtain a representation of the standard Heisenberg algebra, but

$$C_2|\theta_n\rangle = (1-\delta_{n,1})\sqrt{n}\ |\theta_{n-1}\rangle \qquad C_2^\dagger|\theta_n\rangle = \sqrt{n+1}\ |\theta_{n+1}\rangle \tag{5.10}$$

$$[C_2, C_2^\dagger]|\theta_n\rangle = (1+\delta_{n,1})|\theta_n\rangle = \begin{cases} 2|\theta_n\rangle & n=1 \\ |\theta_n\rangle & n \geqslant 2\,. \end{cases} \tag{5.11}$$

This difference is the reason that the two sets of coherent states considered in section 4 are not equal:

$$|z,2\rangle = \frac{r}{\sqrt{\mathrm{e}^{r^2}-1}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(n+1)!}}\,|\theta_{n+1}\rangle \tag{5.12}$$

$$|z,2\rangle_d = \frac{e^{-r^2/2}}{\sqrt{1+r^2}} \sum_{n=0}^{\infty} z^n \sqrt{\frac{(n+1)}{n!}}\ |\theta_{n+1}\rangle\,. \tag{5.13}$$

The measure functions and kernels are also different for the two families:

$$\sigma(r,2) = \frac{1-e^{-r^2}}{\pi} \qquad \sigma_d(r,2) = \frac{1}{\pi}\, e^{r^2}\,(1+r^2)\,E_1(r^2) \tag{5.14}$$

$$\langle z,2|z',2\rangle = e^{i(\varphi-\varphi')}\ \frac{e^{\bar{z}z'}-1}{\sqrt{(e^{r^2}-1)(e^{r'^2}-1)}} \tag{5.15}$$

$$_d\langle z,2|z',2\rangle_d = \frac{(1+\bar{z}z')}{\sqrt{(1+r^2)(1+r'^2)}}\ \exp\left(-\frac{r^2}{2} - \frac{r'^2}{2} + \bar{z}z'\right) \tag{5.16}$$

where $E_1(x)$ is the exponential integral function. As we could expect, the mean values of H_λ in both sets are not equal:

$$\langle z,2|H_\lambda|z,2\rangle = \tfrac{1}{2} + \frac{r^2}{1-e^{-r^2}} \tag{5.17}$$

$$_d\langle z,2|H_\lambda|z,2\rangle_d = \tfrac{3}{2} + \frac{2+r^2}{1+r^2}\ r^2\,. \tag{5.18}$$

Finally, it turns out that the uncertainties of X_2, P_2 and their product are distinct on both sets:

$$(\Delta X_2)^2 = (\Delta P_2)^2 = (\Delta X_2)(\Delta P_2) = \frac{1}{2}\left(1+\frac{r^2}{e^{r^2}-1}\right) \tag{5.19}$$

$$(\Delta X_2)_d^2 = \frac{1}{2(1+r^2)}\left[3+r^2-e^{-r^2}-\frac{(\bar{z}+z)^2}{1+r^2}\right] \tag{5.20}$$

$$(\Delta P_2)_d^2 = \frac{1}{2(1+r^2)}\left[3+r^2-e^{-r^2}+\frac{(\bar{z}-z)^2}{1+r^2}\right] \tag{5.21}$$

$$(\Delta X_2)_d(\Delta P_2)_d = \frac{1}{2(1+r^2)}\left[\left(3+r^2-e^{-r^2}\right)\left(\frac{3+r^4}{1+r^2}-e^{-r^2}\right)-\left(\frac{\bar{z}^2-z^2}{1+r^2}\right)^2\right]^{1/2}. \tag{5.22}$$

Let us notice, once again, that the uncertainties of X_2 and P_2 on the coherent states $|z,2\rangle_d$ do not have radial symmetry (see equations (5.20)–(5.22)). Then, for these states it is possible to design a prescription to induce a maximum efficiency squeezing operation by means of the natural evolution of the system, as discussed at the end of section 4.

Until now, all our results are concerned with the intrinsic structure of H_λ and the distorted coordinate and momentum operators appropiate to this structure. Nevertheless, it would be interesting to find the dispersion for the standard coordinate X and momentum P in the coherent states here derived. This is hard to do for w and λ arbitrary, but can be easily performed in the harmonic oscillator limit, and for particular values of w. We will restrict ourselves to the case $|\lambda| \to \infty$ and $w = 2$. This is justified because, in this limit, C_w and $C_w^\dagger$ behave on $\mathcal{H}_r$ almost exactly as the usual annihilation and creation operators do on $\mathcal{H}$:

$$C_{2,0}|\psi_n\rangle \equiv \lim_{|\lambda|\to\infty} C_2|\theta_n\rangle = \begin{cases} 0 & n=1 \\ a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle & n \geqslant 2 \end{cases} \tag{5.23}$$

$$C_{2,0}^{\dagger}|\psi_n\rangle \equiv \lim_{|\lambda|\to\infty} C_2^{\dagger}|\theta_n\rangle = a^{\dagger}|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle \qquad n \geqslant 1\,. \tag{5.24}$$

All the results derived for X_2, P_2, $|z,2\rangle$ and $|z,2\rangle_d$ remain valid in this limit, where we denote by $|z,2\rangle_0$ and $|z,2\rangle_{d0}$ the coherent states after the limit, and $(\Delta X_2)_0$, $(\Delta X_2)_{d0}$ the uncertainties of X_2 on both sets with the same notation for any other operator. The uncertainties we will obtain for X and P will be compared with those for X_2 and P_2 in (5.19)–(5.22).

First, the mean values of $X = (a^{\dagger}+a)/\sqrt{2}$ and $P = i(a^{\dagger}-a)/\sqrt{2}$ in the states $|z,2\rangle_0$ are

$$_0\langle z,2|X|z,2\rangle_0 = \frac{\bar{z}+z}{\sqrt{2}} \qquad _0\langle z,2|P|z,2\rangle_0 = \mathrm{i}\,\frac{\bar{z}-z}{\sqrt{2}}. \tag{5.25}$$

We also evaluate

$$_0\langle z,2|aa^{\dagger}+a^{\dagger}a|z,2\rangle_0 = 1 + \frac{2r^2}{1-\mathrm{e}^{-r^2}} \tag{5.26}$$

$$_0\langle z,2|a^2|z,2\rangle_0 = \overline{_0\langle z,2|a^{\dagger 2}|z,2\rangle_0} = z^2\,. \tag{5.27}$$

Hence,

$$_0\langle z,2|X^2|z,2\rangle_0 = \frac{1}{2}\left(1 + z^2 + \bar{z}^2 + \frac{2r^2}{1-\mathrm{e}^{-r^2}}\right) \tag{5.28}$$

$$_0\langle z,2|P^2|z,2\rangle_0 = \frac{1}{2}\left(1 - z^2 - \bar{z}^2 + \frac{2r^2}{1-\mathrm{e}^{-r^2}}\right)\,. \tag{5.29}$$

We get now the dispersion of X and P and their product:

$$(\Delta X)_0^2 = (\Delta P)_0^2 = (\Delta X)_0(\Delta P)_0 = \frac{1}{2} + \frac{r^2}{\mathrm{e}^{r^2}-1}\,. \tag{5.30}$$

For $|z,2\rangle_{d0}$ we have

$$_{d0}\langle z,2|X|z,2\rangle_{d0} = \left(\frac{2+r^2}{1+r^2}\right)\frac{(\bar{z}+z)}{\sqrt{2}} \tag{5.31}$$

$$_{d0}\langle z,2|P|z,2\rangle_{d0} = \left(\frac{2+r^2}{1+r^2}\right)\frac{\mathrm{i}(\bar{z}-z)}{\sqrt{2}}\,. \tag{5.32}$$

In order to easily evaluate the deviations of X and P, we first find

$$_{d0}\langle z,2|aa^{\dagger}+a^{\dagger}a|z,2\rangle_{d0} = \frac{(r^2+3)(2r^2+1)}{1+r^2} \tag{5.33}$$

$$_{d0}\langle z,2|a^2|z,2\rangle_{d0} = \overline{_{d0}\langle z,2|a^{\dagger 2}|z,2\rangle_{d0}} = \left(\frac{3+r^2}{1+r^2}\right)z^2\,. \tag{5.34}$$

Hence,

$$_{d0}\langle z,2|X^2|z,2\rangle_{d0} = \frac{1}{2}\left(\frac{r^2+3}{r^2+1}\right)\left[(\bar{z}+z)^2+1\right] \tag{5.35}$$

$$_{d0}\langle z,2|P^2|z,2\rangle_{d0} = \frac{1}{2}\left(\frac{r^2+3}{r^2+1}\right)\left[-(\bar{z}-z)^2+1\right]\,. \tag{5.36}$$

Using the previous results, the dispersions of X and P are given by

$$(\Delta X)_{d0}^2 = \frac{1}{2}\left(\frac{r^2+3}{r^2+1}\right) - \frac{1}{2}\left(\frac{\bar{z}+z}{r^2+1}\right)^2 \tag{5.37}$$

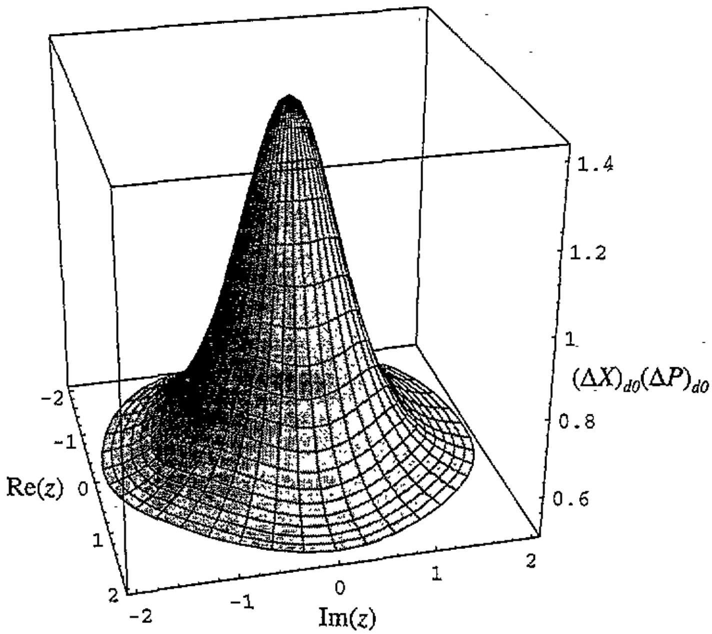

Figure 3. Plot of the uncertainty product for the position X and momentum P operators as a function of z in the harmonic oscillator limit for the coherent states $|z,2\rangle_d$.

and

$$(\Delta P)^2_{d0} = \frac{1}{2}\left(\frac{r^2+3}{r^2+1}\right) + \frac{1}{2}\left(\frac{\bar{z}-z}{r^2+1}\right)^2 . \tag{5.38}$$

A plot of $[(\Delta X)_{d0}(\Delta P)_{d0}]$ as a function of z is shown in figure 3. By comparing equations (5.30) and (5.37), (5.38) with (5.19)–(5.21), we realize that both results are different. The reason is that the subspace $\mathcal{H}_r$, which is invariant under X_2, P_2, is not invariant under X, P. This fact induces additional terms in the formulae, which produce the final difference between the deviations for X, P and for X_2, P_2.

6. Concluding remarks

We have been able to answer the two questions posed in the introduction: there exist creation and annihilation operators for H_λ, the ones with $w=0$ and $w=1$, which behave as the generators of the Heisenberg algebra if their action is rectricted to the invariant subspaces $\mathcal{H}_s$ and $\mathcal{H}_r$, respectively; the two sets of coherent states associated to each one of these operators became essentially equal to the standard coherent states of the harmonic oscillator. It is important to note that these coherent states turned out to be minimum uncertainty states for the distorted coordinate and momentum operators; therefore, we were able to construct indirectly the coherent states described in the third definition. If we had restricted our considerations just to answering those questions, we would never have found the rich family of annihilation and creation operators of the distorted Heisenberg algebra characteristic of H_λ. Moreover, we would never have found the coherent states $|z,w\rangle_d$ on which it is possible to induce the maximum efficiency squeezing operation.

We would like to end this paper with a short comment concerning the widely discussed geometric phase [15, 16]. It arises for any state $|\psi(t)\rangle$ which evolves in a cyclic way during a time interval $[0, \tau]$, i.e. $|\psi(\tau)\rangle = e^{i\phi}|\psi(0)\rangle$, $\phi \in \mathbb{R}$. The phase ϕ can be decomposed as a dynamic plus a geometric part, the last one named geometric phase and denoted β. If the evolution is induced by the Hamiltonian $H(t)$, it turns out that β takes the form [15] ($\hbar = 1$):

$$\beta = \phi + \int_0^\tau \langle\psi(t)|H(t)|\psi(t)\rangle\, dt\,. \tag{6.1}$$

For a time-independent Hamiltonian with equally spaced discrete spectrum, any initial state evolves cyclically [17]. This is true for our family of Hamiltonians (2.5) and (2.6) and the coherent states of section 4. They are cyclic with period $\tau = 2\pi$, $U(2\pi)|z, w\rangle = e^{-i3\pi}|z, w\rangle$ and $U(2\pi)|z, w\rangle_d = e^{-i3\pi}|z, w\rangle_d$. A direct calculation leads us to the following expressions for the geometric phases:

$$\beta = 2\pi\left(\langle H_\lambda\rangle - \tfrac{3}{2}\right) = 2\pi\left(\frac{{}_1F_1(2, w; r^2)}{{}_1F_1(1, w; r^2)} - 1\right) \tag{6.2}$$

$$\beta_d = 2\pi\left(\langle H_\lambda\rangle_d - \tfrac{3}{2}\right) = 2\pi\, r^2 S(w, r^2)\,. \tag{6.3}$$

Hence, $\langle H_\lambda\rangle$ and $\langle H_\lambda\rangle_d$ are, essentially, geometric quantities in the same sense as β and β_d are geometric [17].

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